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Bounded Solutions of a Nonlinear System of Differential-Delay Equations*

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1. INTRODUCTION

This paper is concerned with systems of nonlinear differential-delay equations of the form

$$\begin{aligned} x_i'(t) &= F_i\{\mathbf{x}(t-\tau)\}\, x_i(t), && 1 \leqslant i \leqslant n \\ x_i(t) &\equiv \phi_i(t), && t_0 - \tau \leqslant t \leqslant t_0, \end{aligned} \tag{1}$$

where $\mathbf{x}(t)$ is an n-dimensional vector function of t, with components $x_i(t)$. Most of the results below are obtained by making the further assumption that the functions F_i are linear in the variables $x_1(t-\tau), x_2(t-\tau), \ldots, x_n(t-\tau)$, so that for some real constants a_i and m_{ij} the system can be written

$$\begin{aligned} x_i'(t) &= \left[a_i - \sum_{j=1}^{n} m_{ij} x_j(t-\tau)\right] x_i(t), && 1 \leqslant i \leqslant n \\ x_i(t) &\equiv \phi_i(t), && t_0 - \tau \leqslant t \leqslant t_0. \end{aligned} \tag{2}$$

It is then shown that under certain restrictions on the parameters a_i and m_{ij} all solutions of (2) satisfying $x_i(t_0) > 0$, $1 \leqslant i \leqslant n$, remain positive and uniformly bounded for $t > t_0$. Furthermore, if there is no constant solution of (2) in the positive quadrant of R^n, it is shown that every positive bounded solution approaches the boundary of the positive quadrant as $t \to \infty$.

It is well known [1] that more general systems of differential-delay equations, for example,

$$\begin{aligned} x_i'(t) &= G_i\{t, \mathbf{x}(t), \mathbf{x}(t-\tau)\}, && 1 \leqslant i \leqslant n \\ x_i(t) &\equiv \phi_i(t), && t_0 - \tau \leqslant t \leqslant t_0, \end{aligned}$$

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possess unique solutions for $t > t_0$, if the functions G_i satisfy appropriate Lipschitz conditions. In the case of (1), the system can be integrated once to yield

$$x_i(t) = x_i(t_0) \exp \left\{ \int_{t_0}^{t} F_i[\mathbf{x}(s - \tau)]\, ds \right\}, \qquad 1 \leqslant i \leqslant n. \tag{3}$$

Hence, if the F_i are arbitrary integrable functionals on R^n, the uniqueness and constancy of sign of the $x_i(t)$ follow, using the methods of ordinary differential equations and continuation from the initial interval $[t_0 - \tau, t_0]$ to succeeding intervals of length τ.

In this paper we will restrict our discussion to solutions of (2) in the positive quadrant of R^n. These will subsequently be referred to as *positive solutions*. This choice of sign is motivated by physical considerations.

A system such as (1) can be thought of as representing the growth of n interacting variables, each of which has a relative growth rate x_i'/x_i at time t that depends on the value of all n variables at one previous instant of time, $t - \tau$. Examples of such systems are found in many diverse fields, such as control theory, ecology (growth of populations), epidemology, and in attempts to use mathematics in the study of neurology.

If $n = 1$, (2) reduces to the scalar differential-delay equation

$$x'(t) = [a - x(t - \tau)]\, x(t)$$

studied by Wright [2], Kakutani and Markus [3] and G.S. Jones [4]. If $a > 0$, Kakutani and Markus proved that every solution of this equation satisfying $x(t_0) > 0$ remains uniformly bounded above and below for all $t > t_0$; more precisely, if $M_1 = \max\{x(t)\}$ and $m_1 = \min\{x(t)\}$ for $t_0 \leqslant t \leqslant t_0 + 2\tau$, then

$$0 < m \leqslant x(t) \leqslant M < \infty \qquad \text{on} \qquad t_0 \leqslant t < \infty,$$

where $M = \max\{M_1, ae^{a\tau}\}$ and $m = \min\{m_1, ae^{(a-M)\tau}\}$. It has also been shown [4] that a positive solution $x(t)$ either approaches a asymptotically as $t \to \infty$ or oscillates about a.

After completing this paper, a University of Maryland report by G.M. Dunkel [5] was brought to my attention. In Chapter 3 of this report, Dunkel considers systems identical to (2) above. His results are directed mainly at two-dimensional systems, and complement some of the results reported here. In particular he presents a theorem on the oscillatory behavior of the solutions as $t \to \infty$.

In regard to stability of constant solutions of (2), Dunkel gives partial results for $n = 2$. The results for arbitrary dimension n are contained in [7].

2. Positive Connection Matrix

The matrix $\mathbf{M} = (m_{ij})$ of coefficients in (2) will be called the *connection matrix* of the system. We will assume that a connection matrix satisfies the condition that each $m_{ii} \neq 0$, so that the system of equations can be normalized by the change of variables $y_i = m_{ii}x_i$. If this is done, the matrix of the transformed system will have each diagonal element equal to unity. Two systems (2) will be called *equivalent* if they reduce to identical systems when normalized in this manner. If the connection matrix satisfies $m_{ii} = 1$, $1 \leqslant i \leqslant n$, and if $m_{ij} \geqslant 0$ for all $i \neq j$, $1 \leqslant i, j \leqslant n$, then $\mathbf{M}$ will be called a *positive connection matrix*. The following theorem is an exact analog of the one proved by Kakutani and Markus for the scalar equation [3], and the proof is contained in [5] and [7].

THEOREM 1. *Let* $\mathbf{x}(t)$ *be a positive solution of a system* (2) *with positive connection matrix* $\mathbf{M}$. *Then each component of* $\mathbf{x}$ *satisfies*

$$0 < x_i(t) \leqslant B_i = \max\{a_i e^{a_i \tau}, \max_{t_0 \leqslant t \leqslant t_0 + 2\tau} [x_i(t)]\}.$$

THEOREM 2. *If the connection matrix in* (2) *satisfies* $m_{ii} = 1$, $m_{ij} \geqslant 0$ *for* $j > i$ $(1 \leqslant i \leqslant n)$, *then every positive solution of* (2) *is uniformly bounded on* $t_0 \leqslant t < \infty$.

Proof. Consider the system equations one at a time. For $i = 1$,

$$x_1'(t) = [a_1 - x_1(t - \tau) - m_{12}x_2(t - \tau) - \cdots - m_{1n}x_n(t - \tau)]\, x_1(t)$$

and by the method of proof for Theorem 1, $x_1(t)$ must satisfy

$$0 < x_1(t) \leqslant \max\{a_1 e^{a_1 \tau}, \max_{t_0 \leqslant t \leqslant t_0 + 2\tau} [x_1(t)]\} = B_1 .$$

The second equation can be written

$$\begin{aligned} x_2'(t) = [a_2 \pm | m_{21} |\, x_1(t - \tau) - x_2(t - \tau) - m_{23}x_3(t - \tau) - \cdots \\ - m_{2n}x_n(t - \tau)]\, x_2(t). \end{aligned}$$

Theorem 1 again implies a bound:

$$0 < x_2(t) \leqslant \max\{(a_2 + B_1 | m_{21} |)\, e^{(a_2 + B_1 | m_{21} |)\tau}, \max_{t_0 \leqslant t \leqslant t_0 + 2\tau} [x_2(t)]\} = B_2 .$$

Continuing by induction, each component $x_j(t)$ has a finite bound, $1 \leqslant j \leqslant n$.

It will be shown in Section 5 that if the dimension of the system is greater than or equal to 2 we cannot hope to bound the x_i uniformly away from 0. In fact, we will show that under certain conditions on the parameters, some of the x_i must approach 0 as $t \to \infty$.

3. Closed Feedback Loops

The physical significance of Theorem 2 is seen more clearly if one considers (2) as a system of variables related to each other through delayed feedback. The rate of growth of the variable x_i is given by

$$\frac{x_i'(t)}{x_i(t)} = a_i - m_{i1}x_1(t-\tau) - m_{i2}x_2(t-\tau) - \cdots - m_{in}x_n(t-\tau),$$

so that the effect of x_k on the growth of the variable x_i is determined by the parameter m_{ik}. Negative values of the m_{ik} correspond to positive feedback, with delay, among the system variables.

Definition 1. A *closed feedback loop* in a system (2) is any closed path from an element with index i_1 back to itself, where a path from i_1 to i_n is a sequence

$$m_{i_2 i_1}, m_{i_3 i_2}, \ldots, m_{i_n i_{n-1}}$$

such that each $m_{jk} \neq 0$.

Definition 2. A *reinforcing closed feedback loop* in (2) is a closed feedback loop such that each m_{jk} in the sequence is strictly negative.

In physical terms, a reinforcing closed feedback loop corresponds to a system with positive feedback from an element x_i to itself through a succession of steps which effectively represents a time delay.

Theorem 3. *In a system* (2) *in which there exist no reinforcing closed feedback loops, every positive solution is uniformly bounded on* $t_0 \leqslant t < \infty$.

Proof. Assume no reinforcing closed feedback loop exists. Then there is at least one row of the matrix $\mathbf{M}$ containing all nonnegative elements. For if this were not true, then there would exist a negative element in row 1, say m_{1i_1} ($i_1 \neq 1$ by the definition of a connection matrix). In row i_1 there exists a negative element $m_{i_1 i_2}$ ($i_2 \neq i_1$). Proceeding in this manner, before exhausting all n rows of $\mathbf{M}$ we will have obtained a chain $m_{1i_1}, m_{i_1 i_2} \cdots, m_{i_{k-1} i_k}$ where i_k must be one of the indices $1, i_1, i_2, \ldots, i_{k-2}$, and each m_{pq} in the chain is negative. This chain, written in reverse order, contains a reinforcing closed feedback loop.

Next, interchange the variables so that the all non-negative row is row 1. By considering the $(n-1)$-dimensional submatrix of $\mathbf{M}$ with the 1st row and column deleted, we can find a row i_2 with all nonnegative elements (except possibly the element in the deleted first column). Continuing in this manner, it is clear that the variables in the system can be reordered so that the hypotheses of Theorem 2 are satisfied. Hence the x_i are uniformly bounded.

4. Sufficient Conditions for Boundedness with Negative Elements in M

The conditions in Theorems 2 and 3 are certainly not necessary for boundedness. It is possible to show that if the matrix $\mathbf{M}$ is in some sense "close" to the identity matrix $\mathbf{I}$, the nondiagonal elements can be of arbitrary sign. (On the other hand, it seems very difficult to establish boundedness for systems with negative m_{ij} which are large relative to the diagonal elements m_{ii}.) We denote by $|\mathbf{T}|$ the matrix norm

$$|\mathbf{T}| = \sup_{\mathbf{x}\neq 0} \frac{\|\mathbf{Tx}\|}{\|\mathbf{x}\|},$$

where

$$\|\mathbf{x}\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

Theorem 4. *Let the coefficients of a system* (2) *satisfy the following conditions*

(i) $\mathbf{M} = (m_{ij}) = \mathbf{I} + \mathbf{T}$, *where* $|\mathbf{T}| < 1$;

(ii) $\alpha \equiv \min_{1\leqslant i\leqslant n}(a_i) - \max_{1\leqslant i\leqslant n}(a_i)$ *satisfies* $e^{\alpha\tau} > |\mathbf{T}|$;

(iii) *for* $A = \max_i (a_i)$, $p = \max_{i,j,k} |m_{ij} - m_{kj}|$, *there exists a number* $B^* > 0$ *such that* $B^*(e^{[\alpha - B^*]\tau} - |\mathbf{T}|) > Anpe^{A\tau}$.

Let B be the l.u.b. of the set of real numbers B^* satisfying (iii). *Then all positive solutions of* (2) *satisfying*

$$0 < \sum_{i=1}^{n} x_i(t) < \frac{B}{p} \qquad \textit{for} \qquad t_0 - \tau \leqslant t \leqslant t_0 + \tau$$

remain for all $t > t_0 + \tau$ in the region of the positive quadrant described by $0 < N(t) < B/p$, where $N(t) \equiv \sum_{i=1}^{n} x_i(t)$.

Proof. Since each $x_i(t) > 0$ for all $t \geqslant t_0 - \tau$, $N(t)$ can be considered as a norm for the vector $\mathbf{x}(t)$. Then by hypothesis

$$\begin{aligned} N'(t) = \sum_{i=1}^{n} x_i'(t) &= \sum_{i=1}^{n} a_i x_i(t) - \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} x_i(t)\, x_j(t-\tau) \\ &\leqslant AN(t) - \langle \mathbf{M}\mathbf{x}(t-\tau), \mathbf{x}(t) \rangle \end{aligned} \tag{4}$$

where $\langle \mathbf{u}, \mathbf{v} \rangle$ denotes the ordinary vector inner product. Integrating the original equation once, as in Eq. (3), for any $t \geqslant t_0 + \tau$ we have

$$\begin{aligned} x_i(t) &= x_i(t-\tau) \exp\left\{ a_i\tau - \sum_{j=1}^{n} m_{ij} \int_{t-\tau}^{t} x_j(s-\tau)\, ds \right\} \\ &\equiv x_i(t-\tau)\, e_i(t-\tau). \end{aligned}$$

Let $\mathbf{E}(t-\tau)$ be the diagonal matrix with diagonal elements

$$e_i(t-\tau) = \exp\left\{ a_i\tau - \sum_{j=1}^{n} m_{ij} \int_{t-\tau}^{t} x_j(s-\tau)\, ds \right\};$$

then we can write

$$\mathbf{x}(t) = \mathbf{E}(t-\tau)\, \mathbf{x}(t-\tau).$$

The inner product $\langle \mathbf{M}\mathbf{x}(t-\tau), \mathbf{x}(t) \rangle$ can be written as

$$\langle \mathbf{E}(t-\tau)\, \mathbf{M}\mathbf{x}(t-\tau), \mathbf{x}(t-\tau) \rangle$$

where $\mathbf{M} = \mathbf{I} + \mathbf{T}$. Thus

$$\begin{aligned} &\langle \mathbf{M}\mathbf{x}(t-\tau), \mathbf{x}(t) \rangle \\ &= \langle \mathbf{E}(t-\tau)\, \mathbf{x}(t-\tau), \mathbf{x}(t-\tau) \rangle + \langle \mathbf{E}(t-\tau)\, \mathbf{T}\mathbf{x}(t-\tau), \mathbf{x}(t-\tau) \rangle \\ &\geqslant \{\min_i [e_i(t-\tau)] - \max_i [e_i(t-\tau)] \mid \mathbf{T} \mid\} \, \| \mathbf{x}(t-\tau) \|^2. \end{aligned}$$

Using Eq. (4) and the relationship between $\| \mathbf{x} \|$ and N we obtain the following bound for the derivative of N:

$$N'(t) \leqslant AN(t) - \left[\frac{\min_i [e_i(t-\tau)]}{\max_i [e_i(t-\tau)]} - \mid \mathbf{T} \mid \right] \frac{N(t)\, N(t-\tau)}{n}.$$

For any $t > t_0 + \tau$, the hypotheses of the theorem imply that

$$\frac{\min_i [e_i(t-\tau)]}{\max_i [e_i(t-\tau)]} \geqslant e^{\alpha\tau} e^{-p \int_{t-2\tau}^{t-\tau} N(s)\, ds} \equiv e^{\alpha\tau} e^{-p\tau \overline{N(t-\tau)}}$$

where

$$\overline{N(t-\tau)} = \frac{1}{\tau}\int_{t-2\tau}^{t-\tau} N(s)\,ds$$

is the average of N between $t - 2\tau$ and $t - \tau$. We can then write

$$N'(t) \leqslant \left[A - (e^{\alpha\tau - p\tau\overline{N(t-\tau)}} - |\mathbf{T}|)\frac{N(t-\tau)}{n}\right] N(t).$$

Let $U(t) = pN(t)$. Then

$$U'(t) = pN'(t) \leqslant \left[A - (e^{\alpha\tau - \overline{U(t-\tau)}\tau} - |\mathbf{T}|)\frac{U(t-\tau)}{np}\right] U(t).$$

Since $p > 0$, $U(t)$ is positive for all $t \geqslant t_0 - \tau$. If

$$e^{\alpha\tau - \overline{U(t-\tau)}\tau} > |\mathbf{T}|,$$

then $U'(t) \leqslant AU(t)$.

Now assume B^* satisfies condition (iii), and $N(t) < B^*/p$ for $t_0 - \tau \leqslant t \leqslant t^*$, but $N(t^*) = B^*/p$, that is $U(t^*) = B^*$ for some $t^* > t_0 + \tau$. Then condition (iii) implies that

$$e^{\alpha\tau - \overline{U(t-\tau)}\tau} - |\mathbf{T}| > 0$$

for $t_0 - \tau \leqslant t < t^*$ so that $U'(t) \leqslant AU(t)$. Therefore

$$B^* e^{-A\tau} \leqslant U(t^* - \tau) \leqslant B^*$$
$$B^* e^{-2A\tau} \leqslant U(t^* - 2\tau) \leqslant B^*.$$

This implies that

$$\overline{U(t-\tau)} = \frac{1}{\tau}\int_{t-2\tau}^{t-\tau} U(s)\,ds \leqslant B^*$$

and therefore

$$U'(t^*) \leqslant \left[A - (e^{\alpha\tau - \tau B^*} - |\mathbf{T}|)\frac{B^* e^{-A\tau}}{pn}\right] U(t^*)$$
$$\leqslant \left[\frac{Apne^{A\tau} - B^*(e^{[\alpha - B^*]\tau} - |\mathbf{T}|)}{e^{A\tau}pn}\right] U(t^*) < 0.$$

But U takes on its maximum from the left at $t = t^*$; therefore $U'(t^*) \geqslant 0$. This contradicts the assumption that such a t^* exists; therefore

$$N(t) = \sum_{i=1}^{n} x_i(t) < \frac{B^*}{p} \qquad \text{for all} \qquad t > t_0 - \tau.$$

Notice that the proof of this theorem requires a bound on the $x_i(t)$ over an initial interval of length 2τ. Given bounds on the initial functions $\phi_i(t)$ it is an easy matter to determine the maximum possible increase in the x_i over the interval $t_0 \leqslant t \leqslant t_0 + \tau$. Hence there is no loss of generality in the statement of the theorem.

5. Asymptotic Behavior of Systems with No Positive Constant Solutions

We will define a *semiorbit* $\mathscr{C}^+$ of (2) to be the curve described in n-space by the trajectory of a solution $\mathbf{x}(t)$, $t \geqslant t_0$. A *limit point* $\hat{\mathbf{x}}$ of $\mathscr{C}^+$ will be a point in R^n for which there exists a sequence $\{t_k\}$ of real numbers satisfying $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} \| \mathbf{x}(t_k) - \hat{\mathbf{x}} \| = 0$. The set of all limit points of $\mathscr{C}^+$ will be denoted by $\mathscr{L}(\mathscr{C}^+)$.

The constant solutions of (2) are vectors $\mathbf{x}^*$, with components x_i^* satisfying

$$0 = \left[a_i - \sum_{j=1}^{n} m_{ij} x_j^* \right] x_i^*, \qquad 1 \leqslant i \leqslant n.$$

The vector $\mathbf{x}^* = 0$ is always a solution, and other constant solutions with one or more zero components may also exist. If $\mathbf{x}^*$ is a constant solution, and $x_i^* \neq 0$, $1 \leqslant i \leqslant n$, then $\mathbf{x}^*$ must satisfy

$$\mathbf{M}\mathbf{x}^* = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \equiv \mathbf{A}.$$

The study of the stability of constant solutions is contained in another paper [6].

A vector $\mathbf{x}$ will be called a positive vector if $x_i \geqslant 0$ for $1 \leqslant i \leqslant n$. We will now prove that if $\mathbf{M}\mathbf{x}^* = \mathbf{A}$ has no positive vector as solution, then any bounded positive solution of (2) has the following distinctive asymptotic behavior as $t \to \infty$.

Theorem 5. *If a semiorbit $\mathscr{C}^+$ of* (2), *corresponding to a positive solution, remains uniformly bounded for $t \geqslant t_0$, hence lies in some open hypercube*

$$D = \{(x_1, x_2, \ldots, x_n) \mid 0 < x_i < R_i,\ 1 \leqslant i \leqslant n\},$$

and if $\mathbf{M}\mathbf{x}^* \neq \mathbf{A}$ for any positive vector $\mathbf{x}^*$; then $\mathscr{C}^+$ never intersects itself and the limit set $\mathscr{L}(\mathscr{C}^+)$ is contained in the boundary of D. In fact every limit point $\hat{\mathbf{x}}$ of $\mathscr{C}^+$ has at least one component $\hat{x}_i = 0$.

Proof. If the trajectory $\mathbf{x}(t)$ intersects itself for some $t > t_0$, then there exists a $T > 0$ and $t_1 > t_0$ such that

$$\mathbf{x}(t_1 + T) = \mathbf{x}(t_1).$$

We can write

$$x_i(t_1 + T) = x_i(t_1) \exp\left\{\int_{t_1}^{t_1+T} \left[a_i - \sum_{j=1}^{n} m_{ij} x_j(s - \tau)\right] ds\right\}, \qquad 1 \leqslant i \leqslant n.$$

Since $x_i(t_1) > 0$ for any $t_1 > t_0$, we have

$$x_i(t_1 + T) = x_i(t_1) \Leftrightarrow \int_{t_1}^{t_1+T} \left[a_i - \sum_{j=1}^{n} m_{ij} x_j(s - \tau)\right] ds = 0, \qquad 1 \leqslant i \leqslant n.$$

This is a system of linear equations

$$\mathbf{M} \cdot \begin{bmatrix} \dfrac{1}{T} \displaystyle\int_{t_1}^{t_1+T} x_1(s - \tau)\, ds \\ \vdots \\ \dfrac{1}{T} \displaystyle\int_{t_1}^{t_1+T} x_n(s - \tau)\, ds \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

By hypothesis, $\mathbf{Mx}^* = \mathbf{A}$ has no positive solutions, hence this equation cannot be satisfied for any $T > 0$.

Now assume that there exists a limit point $\hat{\mathbf{x}} \in \mathscr{L}(\mathscr{C}^+)$ such that each component $\hat{x}_i > 0$. Let $\min_{1 \leqslant i \leqslant n} (\hat{x}_i) = \delta > 0$. Since $\hat{\mathbf{x}} \in \mathscr{L}(\mathscr{C}^+)$, there exists a sequence of real numbers $\{t_k\}$, each $t_k > t_0$, with $\lim_{k \to \infty} t_k = \infty$ and such that $\mathbf{x}(t_k) \to \hat{\mathbf{x}}$ as $k \to \infty$. We will select a subsequence of $\{t_k\}$ such that each pair t_k, t_{k+1} satisfies $t_{k+1} - t_k > 1$. There exists a spherical neighborhood $N(\hat{\mathbf{x}}, \delta/2)$, with center at $\hat{\mathbf{x}}$ and radius $\delta/2$, which lies entirely in the open hypercube D. Since $\hat{\mathbf{x}}$ is a limit point of $\mathscr{C}^+$, we can also pick the sequence $\{t_k\}$ so that $\mathbf{x}(t_k) \in N(\hat{\mathbf{x}}, \delta/2)$ for all $k = 1, 2, \ldots$. Assume that this has been done. Then for each k,

$$\min_{1 \leqslant i \leqslant n} x_i(t_k) \geqslant \frac{\delta}{2}.$$

We now write

$$x_i(t_k) = x_i(t_{k-1}) \exp\left\{\int_{t_{k-1}}^{t_k} \left[a_i - \sum_{j=1}^{n} m_{ij} x_j(s - \tau)\right] ds\right\}.$$

Then

$$\| \mathbf{x}(t_k) - \mathbf{x}(t_{k-1}) \|^2$$

$$= \sum_{i=1}^{n} | x_i(t_k) - x_i(t_{k-1})|^2$$

$$= \sum_{i=1}^{n} | x_i(t_{k-1})|^2 \left| \exp \left\{ \int_{t_{k-1}}^{t_k} \left[a_i - \sum_{j=1}^{n} m_{ij} x_j(s - \tau) \right] ds \right\} - 1 \right|^2$$

$$\geqslant \frac{\delta^2}{4} \sum_{i=1}^{n} \left| \exp \left\{ a_i(t_k - t_{k-1}) - \sum_{j=1}^{n} m_{ij} \int_{t_{k-1}}^{t_k} x_j(s - \tau)\, ds \right\} - 1 \right|^2$$

$$= \frac{\delta^2}{4} \sum_{i=1}^{n} \left| \exp \left\{ (t_k - t_{k-1}) \left[a_i - \sum_{j=1}^{n} \frac{m_{ij}}{(t_k - t_{k-1})} \int_{t_{k-1}}^{t_k} x_j(s - \tau)\, ds \right] \right\} - 1 \right|^2$$

$$\geqslant \frac{\delta^2}{4} \sum_{i=1}^{n} \left| \exp \left\{ a_i - \sum_{j=1}^{n} \frac{m_{ij}}{(t_k - t_{k-1})} \int_{t_{k-1}}^{t_k} x_j(s - \tau)\, ds \right\} - 1 \right|^2,$$

where the final inequality holds because of the choice of the sequence $\{t_k\}$ and the fact that $| e^{rx} - 1 | > | e^x - 1 |$ for any real number x if $r > 1$.

Now we use the fact that $\| \mathbf{A} - \mathbf{M}\mathbf{v} \| > 0$ for all $\mathbf{v}$ in the closure of D. Therefore the function

$$\sum_{i=1}^{n} \left| \exp \left\{ a_i - \sum_{j=1}^{n} \frac{m_{ij}}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} x_j(s - \tau)\, ds \right\} - 1 \right|^2$$

has an absolute minimum > 0 on this compact set, independent of k. This contradicts the fact that $\mathbf{x}(t_k) \to \hat{\mathbf{x}}$ as $k \to \infty$. Therefore $\mathscr{L}(\mathscr{C}^+)$ consists only of vectors which have at least one component equal to 0.

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